ASYMPTOTIC EXPANSION FOR DISTRIBUTION OF MARKOVIAN RANDOM MOTION

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Abstract. In this paper we study an asymptotic expansion for the distribution of a random motion of a particle driven by a Markov process in diffusion approximation. We show that the singularly perturbed equation of a Markovian random motion can be reduced to the regularly perturbed equation for the distribution of the random motion.

1. Introduction

The first CLT for additive functionals of a Markov chain with noncountable phase space was proved by Doeblin [1]. Additional functionals of Markov and semi-Markov processes with finite phase space have intensively been studied by V.S.Korolyuk, A.F.Turbin, M.Pinsky, V.M.Shurenkov and others [4],[5],[6].

In 1951 S. Goldstein introduced the telegrapher's stochastic process in his seminal paper [2], which is a random motion driven by a homogeneous Poisson process. This basic telegrapher process has been extended by M.Kac in [3]. Goldstein-Kac's telegraph process on the line, and its weak convergence to the one-dimensional Brownian motion is well-known.

This paper deals with the n-dimensional random motion which is an additional functional of some Markov process. This kind of models is well known and popular in the physical literature for the description of the long polymer molecules. For example, one of the forms of the Airing model in [7] is similar to the model in this paper.

Let us consider the random motion of a particle in $\mathbb{R}^n$ driven by a Markov process $\xi(t)$, which sojourn times at states are exponential distributed with rate $\lambda > 0$ and transition probabilities $p_{ij} = \frac{1}{2n-1}\delta_{ij}$, $i, j \in E = \{1, 2, \dots, 2n\}$, where E is the phase space of $\xi(t)$.

Let $\vec{b}_1, \dots, \vec{b}_n$ be a Cartesian basis of $\mathbb{R}^n$. Put $\vec{e}_1 = \vec{b}_1$, $\vec{e}_2 = -\vec{b}_1$, $\vec{e}_3 = \vec{b}_2$, $\vec{e}_4 = -\vec{b}_2$, ..., $\vec{e}_{2n-1} = \vec{b}_n$, $\vec{e}_{2n} = -\vec{b}_n$ and $\vec{v}_i = v\vec{e}_i$, $i = 1, 2, \dots, 2n$, where $v > 0$ is constant speed of the particle.

We assume that the particle moves in n-dimensional space in the following manner: If at some instant t the particle has velocity $\vec{v}_i$, then at a renewal moment of the Markov process the particle takes a new velocity $\vec{v}_j$, $j \neq i$ with probability $p_{ij} = \frac{1}{2n-1}$. The particle continues its motion with velocity $\vec{v}_j$ until the next renewal moment of the Markov process, and so on.

Let us denote by $\vec{r}(t) = (x_1(t), x_2(t), \dots, x_n(t))$, $t \geq 0$ the particle position at time t. Consider the function

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$$\vec{C}(i) = (C_1(i), C_2(i), \ldots, C_n(i)) = \vec{v}_i,$$

$i \in E$.

Then the position of the particle at time t can be expressed as

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{C}(\xi(t))dt.$$

2. Equation for probability density of particle position

Let us consider the bivariate stochastic process $\varsigma(t) = (\vec{r}(t), \xi(t))$ with the phase space $\mathbb{R}^n \times E$. It is well known that this process is Markovian and the generating operator of $\varsigma(t)$ is of the following form [4],[8]:

$$A\varphi(\vec{r}, i) = \vec{C}(i)\varphi'(\vec{r}, i) + \lambda[P\varphi(\vec{r}, i) - \varphi(\vec{r}, i)], \tag{1}$$

where

$$\vec{C}(i)\varphi'(\vec{r}, i) = C_1(i)\frac{\partial}{\partial x_1}\varphi(\vec{r}, i) + C_2(i)\frac{\partial}{\partial x_2}\varphi(\vec{r}, i) + \ldots + C_n(i)\frac{\partial}{\partial x_n}\varphi(\vec{r}, i),$$

and $P\varphi(\vec{r}, i) = \frac{\lambda}{2n-1}\sum\limits_{j\in E\setminus i}\varphi(\vec{r}, i)$.

Now, let us consider the density function

$$\begin{aligned} f_i(t, x_1, \quad \ldots \quad &, x_n)dx_1 \ldots dx_n = \\ = \quad & P\{x_1 \leq \ x_1(t) \leq x_1 + dx_1, \ldots, x_n \leq \ x_n(t) \leq x_n + dx_n\} \end{aligned}$$

It is easily verified that $f(t, x_1, \ldots, x_n) = \sum\limits_{i=1}^{n} f_i(t, x_1, \ldots, x_n)$ is the probability density of the particle position in $\mathbb{R}^n$ at time t.

Lemma 1. *The function f satisfies the following differential equation*

$$\prod_{i=1}^{2n}\{\frac{\partial}{\partial t} + (-1)^i v\frac{\partial}{\partial x_i} + \frac{2n\lambda}{2n-1}\}f +$$
$$\frac{2n\lambda}{2n-1}\sum_{k=1}^{2n}\prod_{\substack{i=1\\ i\neq k}}^{2n}\{\frac{\partial}{\partial t} + (-1)^i v\frac{\partial}{\partial x_i} + \frac{2n\lambda}{2n-1}\}f = 0$$

Proof. For $i \in E$ function f_i satisfies the first Kolmogorov equation, namely

$$\frac{\partial f_i(t, x_1, \ldots, x_n)}{\partial t} = Af_i(t, x_1, \ldots, x_n), \quad i \in E \tag{2}$$

with initial conditions $f_i(0, x_1, \ldots, x_n) = f_i^{(o)}$.

Eq.(2) can be written in more detail as follows

$$\frac{\partial f_i(t, x_1, \ldots, x_n)}{\partial t} + (-1)^i v\frac{\partial}{\partial x_i}f_i(t, x_1, \ldots, x_n) + \lambda f_i(t, x_1, \ldots, x_n) -$$
$$\frac{\lambda}{2n-1}\sum_{j\in E\setminus i} f_i(t, x_1, \ldots, x_n) = 0, \quad i \in E. \tag{3}$$

Now, put $\vec{f}(t, x_1, \ldots, x_n) = \{f_i(t, x_1, \ldots, x_n), i \in E\}$. The set of equations (3) can be written in the following form

$$L_{2n}\vec{f} = 0,$$

where $L_{2n} = \{l_{ij}\}_{ij\in E}$, $l_{ii} = \frac{\partial}{\partial t} + (-1)^i v\frac{\partial}{\partial x_i} + \lambda$, $l_{ij} = \frac{-\lambda}{2n-1}$, $i \neq j$.

The function f satisfies the following equation [9]

$$\det(L_{2n})f = 0, \tag{4}$$

with the initial condition $f(0, x_1, \ldots, x_n) = \sum_{k=1}^{n} f_i^{(o)}$.

The determinant of matrix L_{2n} is well known and it has the form

$$\det(L_{2n}) = \prod_{i=1}^{2n}\{\frac{\partial}{\partial t} + (-1)^i v\frac{\partial}{\partial x_i} + \frac{2n\lambda}{2n-1}\} +$$
$$\frac{2n\lambda}{2n-1}\sum_{k=1}^{2n}\prod_{\substack{i=1\\ i\neq k}}^{2n}\{\frac{\partial}{\partial t} + (-1)^i v\frac{\partial}{\partial x_i} + \frac{2n\lambda}{2n-1}\}$$

□

Since $v\frac{\partial}{\partial x_i}$ and $-v\frac{\partial}{\partial x_i}$ appear in L_{2n} symmetrically it is easy to see that all monomials of the polynomial $\det(L_{2n})$ contain v^k only with even powers $k \geq 0$.

3. Reduction of singularly perturbed evolution equation to regularly perturbed equation

Let us put $v = \varepsilon^{-1}$ and $\lambda = \varepsilon^{-2}$, where $\varepsilon > 0$ is a small parameter. It is well known [8],[10],[11] that the solution of Eq.(3) in hydrodynamical limit (as $\varepsilon \to 0$) weakly converges to the corresponding functional of Wiener process.

By using technique developed in [13], we can find asymptotic expansion of the solution of Eq.(3), which consists of regular and singular terms. This technique involves tedious calculations [4],[10].

Proposition. *The equation* $\det(L_{2n})f = 0$ *is regular perturbed, that is, multiplying it by* ε^{4n-2}, *we get*

$$\frac{\partial}{\partial t}f = \frac{2n-1}{2n^2}(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2})f + D_\varepsilon f,$$

where $D_\varepsilon = \varepsilon^2 D_1 + \varepsilon^4 D_2 + \ldots$, D_i, $i = 1, 2, \ldots$, *are respective differential operators.*

Proof. To avoid cumbersome expressions, we consider the case when $n = 3$. Let us put $x = x_1$, $y = x_2$, $z = x_3$. In this case Eq.(3) has the following form

$$\frac{\partial f_i(t,x,y,z)}{\partial t} + (-1)^i v\frac{\partial}{\partial x}f_i(t,x,y,z) + \lambda f_i(t,x,y,z) - \frac{\lambda}{5}\sum_{\substack{j\in E\\ j\neq i}} f_i(t,x,y,z), \tag{5}$$

$i = 1, \ldots, 6$.

Putting $v = \varepsilon^{-1}$ and $\lambda = \varepsilon^{-2}$, we obtain the following singularly perturbed system of equations

$$\frac{\partial f_i(t,x,y,z)}{\partial t} + (-1)^i\varepsilon^{-1}\frac{\partial}{\partial x}f_i(t,x,y,z) + \varepsilon^{-2}f_i(t,x,y,z) -$$
$$\varepsilon^{-2}\frac{1}{5}\sum_{\substack{j\in E\\ j\neq i}} f_i(t,x,y,z) = 0, \tag{6}$$

$i = 1, \ldots, n$.

Let us consider the equation $\det(L_6)f = 0$, where $f(t,x,y,z) = \sum\limits_{i=1}^{6} f_i(t,x,y,z)$. It is easy to see that elements of matrix $L_6 = (l_{ij})_{i,j\in E}$ are as follows: $l_{ii} = \frac{\partial}{\partial t} + (-1)^i v\frac{\partial}{\partial t} + \lambda$ and $l_{ij} = \frac{\lambda}{5}$, for $i \neq j$, $i,j \in \{1,2,\dots,6\}$.

Hence, the equation $\det(L_6)f = 0$ has the following form

$$\det(L_6)f(t,x,y,z) = \{\frac{7776}{3125}\varepsilon^{-10}\frac{\partial}{\partial t} - \frac{432}{625}\varepsilon^{-10}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) -$$

$$\frac{432}{125}\varepsilon^{-8}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial}{\partial t} - \frac{144}{25}\varepsilon^{-6}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^2}{\partial t^2} +$$

$$\frac{1296}{125}\varepsilon^{-2}\frac{\partial^2}{\partial t^2} + \frac{24}{25}\varepsilon^{-8}\left(\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^2}{\partial y^2\partial z^2} + \frac{\partial^2}{\partial x^2\partial z^2}\right) + \frac{72}{5}\varepsilon^{-4}\frac{\partial^4}{\partial t^4} -$$

$$4\varepsilon^{-4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^3}{\partial t^3} + 2\varepsilon^{-6}\left(\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^2}{\partial y^2\partial z^2} + \frac{\partial^2}{\partial x^2\partial z^2}\right)\frac{\partial}{\partial t} +$$

$$6\varepsilon^{-2}\frac{\partial^5}{\partial t^5} + \varepsilon^{-8}\left(\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^2}{\partial y^2\partial z^2} + \frac{\partial^2}{\partial x^2\partial z^2}\right)\frac{\partial^2}{\partial t^2} - \varepsilon^{-6}\frac{\partial^6}{\partial x^2\partial y^2\partial z^2} -$$

$$\varepsilon^{-2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^4}{\partial t^4} + \frac{\partial^6}{\partial t^6} + \frac{432}{25}\varepsilon^{-6}\frac{\partial^3}{\partial t^3}\}f(t,x,y,z) = 0, \tag{7}$$

with the initial condition $f(0,x,y,z) = f_0$.

Multiplying Eq.(7) by ε^{10}, we obtain

$$\{\frac{\partial}{\partial t} - \frac{5}{18}\Delta + \varepsilon^2\frac{25}{6}\left[\frac{\partial^2}{\partial t^2} - \frac{1}{3}\Delta\frac{\partial}{\partial t} + \frac{5}{54}\Delta^{(2)}\right] +$$

$$\varepsilon^4\frac{125}{18}\left[\frac{\partial^3}{\partial t^3} - \frac{1}{3}\Delta\frac{\partial^2}{\partial t^2} + \frac{5}{216}\Delta^{(2)}\frac{\partial}{\partial t}\right]$$

$$+\varepsilon^6\frac{625}{216}\left[\frac{\partial^4}{\partial t^4} - \frac{5}{18}\Delta\frac{\partial^3}{\partial t^3} + \frac{5}{72}\Delta^{(2)}\frac{\partial^2}{\partial t^2}\right] +$$

$$\varepsilon^8\frac{3125}{1296}\left[\frac{\partial^5}{\partial t^5} - \frac{1}{6}\Delta^{(2)}\frac{\partial^4}{\partial t^4}\right] + \varepsilon^{10}\frac{3125}{7776}\frac{\partial^6}{\partial t^6}\}f = 0, \tag{8}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $\Delta^{(2)} = \frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^2\partial z^2} + \frac{\partial^4}{\partial x^2\partial z^2}$.

□

Lemma 2. *The solution of equation (8) with the initial condition* $f(0,x,y,z) = u_0(0,x,y,z) + \varepsilon^2 u_1(0,x,y,z) + \varepsilon^4 u_2(0,x,y,z) + \dots$, *has asymptotic expansion*

$$f(t,x,y,z) = u_0(t,x,y,z) + \varepsilon^2 u_1(t,x,y,z) + \varepsilon^4 u_2(t,x,y,z) + \dots, \tag{9}$$

where the principal term $u_0(t,x,y,z)$ *represents the solution of equation*

$$\frac{\partial}{\partial t}u_0(t,x,y,z) = \frac{5}{18}\Delta u_0(t,x,y,z).$$

Proof. To find the asymptotic expansion of the solution of (8) we use the method proposed in [13] and developed in [4]. In conformity with this method the solution of (8) can be expanded into the series (9), where $\varepsilon > 0$ is small.

Substituting (9) into (8), we get the following equations for computing of u_i, $i \geq 0$

$$\frac{\partial}{\partial t}u_0(t,x,y,z) = \frac{5}{18}\Delta u_0(t,x,y,z);$$

$$\frac{\partial}{\partial t}u_1(t,x,y,z) = \frac{5}{18}\Delta u_1(t,x,y,z)+$$
$$\varepsilon^2\frac{25}{6}\left[\frac{\partial^2}{\partial t^2} - \frac{1}{3}\Delta\frac{\partial}{\partial t} + \frac{5}{54}\Delta^{(2)}\right]u_0(t,x,y,z);$$

$$\frac{\partial}{\partial t}u_2(t,x,y,z) = \frac{5}{18}\Delta u_2(t,x,y,z)+$$
$$\varepsilon^2\frac{25}{6}\left[\frac{\partial^2}{\partial t^2} - \frac{1}{3}\Delta\frac{\partial}{\partial t} + \frac{5}{54}\Delta^{(2)}\right]u_1(t,x,y,z)+$$
$$\frac{125}{18}\left[\frac{\partial^3}{\partial t^3} - \frac{1}{3}\Delta\frac{\partial^2}{\partial t^2} + \frac{5}{216}\Delta^{(2)}\frac{\partial}{\partial t}\right]u_0(t,x,y,z);$$

$$\dots$$

$$\frac{\partial}{\partial t}u_{m+5}(t,x,y,z) = \frac{5}{18}\Delta u_{m+5}(t,x,y,z)+$$
$$\frac{25}{6}\left[\frac{\partial^2}{\partial t^2} - \frac{1}{3}\Delta\frac{\partial}{\partial t} + \frac{5}{54}\Delta^{(2)}\right]u_{m+4}(t,x,y,z)+$$
$$\frac{125}{18}\left[\frac{\partial^3}{\partial t^3} - \frac{1}{3}\Delta\frac{\partial^2}{\partial t^2} + \frac{5}{216}\Delta^{(2)}\frac{\partial}{\partial t}\right]u_{m+3}(t,x,y,z)+$$
$$\frac{625}{216}\left[\frac{\partial^4}{\partial t^4} - \frac{5}{18}\Delta\frac{\partial^3}{\partial t^3} + \frac{5}{72}\Delta^{(2)}\frac{\partial^2}{\partial t^2}\right]u_{m+2}(t,x,y,z)+$$
$$\frac{3125}{1296}\left[\frac{\partial^5}{\partial t^5} - \frac{1}{6}\Delta^{(2)}\frac{\partial^4}{\partial t^4}\right]u_{m+1}(t,x,y,z) + \frac{3125}{7776}\frac{\partial^6}{\partial t^6}u_m(t,x,y,z) = 0,$$

for $m \geq 0$. □

Let us consider the function

$$\widetilde{f}_k^{(\varepsilon)}(t,x,y,z) = u_0(t,x,y,z) + \varepsilon^2 u_1(t,x,y,z) + \ldots + \varepsilon^{2k}u_k(t,x,y,z).$$

In [10] for the solution of a singularly perturbed equation of type (3) the remainder of asymptotic expansion in the circuit of diffusion approximation was studied.

Taking into account that $f(t,x,y,z) = \sum_{i\in E} f_i(t,x,y,z)$, it follows from the estimate of the reminder in [4],[12] that

$$\| f(t,x,y,z) - \widetilde{f}^{(\varepsilon)}(t,x,y,z) \| = O(\varepsilon^{2k})$$

4. Conclusions

Singularly perturbed equations of type (3) in hydrodynamical limit (where $\frac{\lambda}{v^2} = O(1), \lambda \downarrow 0$) have become the subject of a great deal of researches [3],[4],[6],[10],[11] and others. By using the technique of professor A.F.Turbin [9], we reduce singularly perturbed system of equations (3) to Eq.(4), which turns out to be regularly perturbed in hydrodynamical limit to diffusion process.

Therefore, in such cases we may simplify cumbersome calculations of terms of asymptotic expansion for solution of singulary perturbed equations for functionals of Markovian random motion.

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